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Study of Chance Functions

Abstract

L.A. Zadeh takes the first step in developing the principle and basis of possibility theory as a qualitative mathematical concept analogous to probability. A high degree of probability always implies a high degree of possibility but not conversely. Also probabilities can add but this is not the rule for possibilities. The function C_h, which to each proposition assigns corresponding possibility, is the chance function. A set of conditions governing such a chance functions are discussed and the necessary consequences of rational betting are shown in this paper.

Keywords: Probability, Possibility, Possibility Statements, Chance Functions, Probability Measure.

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Introduction

Early in the 1950's decade, the economist G.L.S. Shackle had once proposed and worked on a non-probabilistic model of expectation but his works were not accepted as an alternative to probability theory. In 1978, professor L.A. Zadeh takes the first step in developing the principle and basis of possibility theory as a qualitative mathematical concept analogous to probability. He exemplifies the conversational use of these terms by noting differences between them. For example, a high degree of probability always implies a high degree of possibility but not conversely. Also (in appropriate circumstances) probabilities add, but this is not the rule for possibilities.

To understand the difference between "Probability and Possibility" let us see some simple examples:

Example 1

Let us consider a house with only one door and one window. The possibility of a thief entering the house is the maximum of the probabilities of entering through the door and through the window. The probability of a thief entering the same house is the sum of the probabilities of entering through the door and through the window (since he cannot enter through both). Hence, the possibility and probability of a thief entering the house are clearly different. However, the possibilities of entering through the door and through the window can respectively be interpreted as the probabilities of two random events:

A = {the thief is able to enter through the door}
B = {the thief is able to enter through the window}

Hence, the possibility of a thief entering the house can be interpreted as the maximum of the probabilities of events A and B, or what is the same, as the probability of the most likely of these two events.

Example 2

Consider the following statement:

"Navya ate X mangoes yesterday" with X taking values in N = {1, 2, 3… 8}.

We can associate a possibility distribution C_h (n) with N as the degree of ease Navya can eat n mangoes and we can also associate a probability distribution P (n) as she is eating a day. Then the values of C_h (n) and P (n) can be shown as:

Table1.

N	1	2	3	4	5	6	7	8
C_h (n)	1	1	1	1	0.8	0.6	0.4	0.2
P (n)	0.1	0.8	0.1	0	0	0	0	0

From table 1, we find that the possibility that Navya can eat 3 mangoes is 1 but the probability of the same might be very-very small i.e. 0.1.

Thus, a high degree of possibility does not imply a high degree of probability, nor does a low degree of probability imply a low degree of possibility.

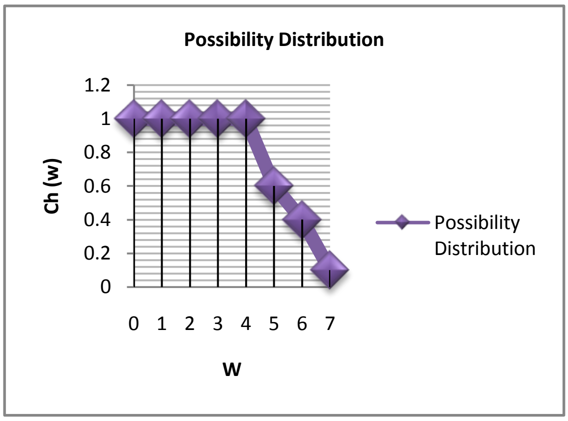
Example 3

Let us take into account F number of goals are admitted by Indian Hockey Team in an international match. We infer that F takes the values in W = {0, 1, 2, 3, ….. , 7}. Then degree of belief C_h (w) with which Indian Hockey Team admits F number of goals and probability P (w) with which Indian Hockey Team will admit F number of goals can be made clear by the following table:

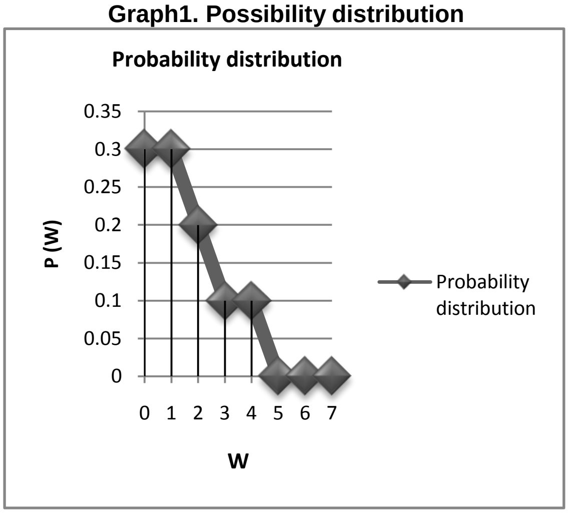
Table2.

W	0	1	2	3	4	5	6	7
C_h (w)	1	1	1	1	1	0.6	0.4	0.1
P (w)	0.3	0.3	0.2	0.1	0.1	0	0	0

This possibility and probability distribution can be exhibited graphically as follows



Graph1. Possibility distribution



Graph2. Probability distribution

It is clear from the table 2 and graph 1 and 2 that high value of possibility does not indicate a corresponding high probability value but rather shows that a probable event is indeed possible and also that an impossible event is indeed possible.

Example 4

Let us consider the statement

S: "Kamal is now in Mumbai."

The person who admits this statement should be willing to pay a penalty of some sort incase it turns out to be false. Practically this penalty would ethically be loss of face, but for theory we need something more definite. For simplicity we take:

Definition 1

The person who makes this assertion will have to pay a penalty of ₹ 1 if the statement turns out to be false i.e. here the commitment is the obligation to pay ₹ 1 as fine in case one is proved wrong.

Suppose Mr. X assert, "In my opinion the probability that Kamal is in Mumbai is 0.8." Then in view of the Definition 1 Mr. X will have to pay a fee of ₹ 0.2 to assert "Kamal is in Mumbai". At the same time he will have to pay ₹ 0.8 to assert "Kamal is not in Mumbai". Reducing it into general terms we have:

Definition 2

The person who makes the assertion "The probability of S is k (where $0 \le k \le 1$) agrees:

1. To assert S for a fee of ₹ (1-k) and
2. To assert ~S for a fee of ₹ k

A very cautious person might well (on grounds of ignorance) refuse to agree to (i) unless k = 0 and at the same time refuse to agree to (ii) unless k = 1, he can hardly be accused of being irrational for adopting this conservative attitude. Thus, the probability function of a person may not be defined for all propositions.

The Interpretation of Possibility Statements

Let us now apply the pragmatic method to statements involving the term "possible". For instance, we interpret "There is slight possibility that Kamal is in Mumbai" as "the probability (in my opinion) that Kamal is in Mumbai does not exceed (say) 0.1".

This assertion in view of Definition 2 indicates willingness for ₹ 0.1 to pay ₹ 1 incase Kamal is in Mumbai. This commitment gives the practical meaning of the possibility assignment and we can adopt:

Definition 3

For any proposition S and any agent the possibility C_h (S) of S is the smallest number k such that for a fee of ₹ k the agent will agree to pay ₹1 if S is found to be the case. The function C_h, which to each proposition assigns the corresponding possibility, is called "possibility function /chance function" of the agent.

Thus, in contrast to the situation for probability, the possibility C_h (S) has, for any rational agent, a value for every proposition S. Here, "S is impossible", interpreted as "the possibility of S is zero" is equivalent to "~S" and "S is entirely possible" entails no possibility of loss. Also, there is a close relation between possibility and probability. For instance, to say "S has probability 0.8" is equivalent to saying both "S has possibility 0.8" and "~S has possibility 0.2".

Properties of Chance Functions

We shall use the pragmatic explanation of a possibility assignment to unfold the formal properties of chance functions. In doing this we will discuss the relations between the possibilities of logically related propositions, e.g. A, not A, B, A and B, A or B, etc. Let us assume that we have a Boolean algebra Ɓ of propositions. A chance function will then be a real valued function defined on this Boolean algebra.

We have the knowledge of the elementary method of representing logical relations among propositions by a Venn diagram: each proposition is represented by a set in such a way that intersections correspond to conjunctions and unions to disjunctions. In fact, this can be done for any Boolean algebra Ɓ: according to Stone Representation Theorem there is a set τ such that Ɓ may be represented as a Boolean algebra of subsets of τ: more precisely, there is a topological space τ (a totally disconnected compact Hausdorff space) such that Ɓ is isomorphic to the Boolean algebra of all closed – open subsets of τ. This representation of Ɓ is very convenient in discussing the properties of chance functions.

For practical purposes we may identify each proposition with a subset of τ. Then, for any propositions A and B, A and B and A or B are the intersection A ∩ B and union A ∪ B respectively and A^c is the complement (in τ) of A; also, $A \Longrightarrow B$ corresponds to the logical relation A implies B. It is convenient to refer to the points of τ as "possible worlds" calling a proposition A "true in a possible world X" exactly if X ∈ A. τ itself then becomes the set of all possible worlds (or the "universe of discourse" in Zadeh's terminology). Regarded as a proposition, τ is the true proposition – it is true in every possible world. Similarly, the complement of τ, the empty set φ, is the false proposition F, the proposition which is false in every possible world. Of course, $F^c = \tau$ and $\tau^c = F$. Two technical points: (a) Beware of assigning any ontological significance to the possible worlds: they arise only through the Stone representation theorem, the term "possible world" being just a figure of speech. (b) We note that in general only certain subsets of τ (the clopen subsets) denote by $\underset{\square}{\mathbf{1}}$ and $\underset{\square}{\mathbf{0}}$ the functions on τ that take the constant values 1 and 0 respectively and, for a proposition denoted by a capital letter, A say, (possibly with subscript) we denote by the corresponding small letter a (with the same subscript) the characteristic function of A, i.e. the function that has value 1 if A is true and zero otherwise. In particular, t = $\underset{\square}{\mathbf{1}}$ and f = $\underset{\square}{\mathbf{0}}$ .Now we assume that C_h is the chance function corresponding to some agent. If the agent is rational this function must have certain properties. We shall state a number of these and justify them (A and B here denote arbitrary propositions):

(1) $C_h(F) = 0, \quad C_h(\tau) = 1.$

(2) $A \Longrightarrow B$ implies $C_h(A) \le C_h(B)$.

These follow directly from definition 3, since F never holds and τ always holds, and if $A \Longrightarrow B$ then B holds whenever A holds.

(3) $\max\left(C_h(A), C_h(B)\right) \le C_h(A \cup B) \le C_h(A) + C_h(B).$

The left inequality follows from (2). For the right inequality s ≥ C_h (A) and t ≥ C_h (B). Then for Rs (s + t) the agent will assert both A and B. He will then be obliged to pay at least ₹ 1 if either A or B holds (in fact ₹ 2 if they both hold). Since this is a greater obligation than that incurred by asserting A ∪ B we must have s + t ≥ C_h (A ∪ B). The right inequality in (3) now follows, since s and t are arbitrary.

(4) If A ∪ B = τ then $C_h(A) + C_h(B) \ge 1$.

This follows immediately from (3) and (1); and as a particular case of (4) we have:

(5) $C_h(A) + C_h(A^c) \ge 1.$

Let $C_h(A) + C_h(A^c) = 1$. Then putting $c = C_h(A)$ and using Definition 3, we see that k = c satisfies (ii) of Definition 2, and similarly applying Definition 3 to $1 - c = C_h(A^c)$ we find k = c satisfies (i) of definition 2. Together this means c (A) exists and c (A) = C_h (A). Similarly, if, in Definition 2, A is replaced by A^c, then k = 1 – c satisfies (ii) and also satisfies (i), which means $c(A^c) = 1 - c = C_h(A^c)$.

Conversely, let c (A) exists. Then, $C_h(A) \le c(A)$ (by definition2 (ii) and Definition (3) and similarly $C_h(A^c) \le c(A^c)$. But, by the symmetry of Definition 2 we have $c(A) = 1 - c(A^c)$. Thus, $C_h(A) \le c(A) = 1 - C_h(A^c) \le 1 - C_h(A^c) \le C_h(A)$, by (5), whence all inequalities become equalities and $C_h(A) = c(A), C_h(A^c) = c(A^c) \& C_h(A) + C_h(A^c) = 1.$ Therefore, we have the following theorem:

Theorem 1

A necessary and sufficient condition that c (A) exists is that $C_h(A) + C_h(A^c) = 1$. If this condition holds then $c(A) = C_h(A)$ and $c(A^c) = C_h(A^c)$.

One might hope that by listing a sufficient number of properties like (1) to (5) one could characterize a chance function in the sense that every function having the listed properties could be shown to be the chance function of some rational agent. The listed properties could then be used as axioms for a mathematical theory of chance functions in complete accord with the semantics given above. In fact, it turns out that one property suffices for this purpose; unfortunately, it is more complicated than those given so far, but it is equally easy to justify:

(6) Let n, r, s are non-negative integers and A_0, A_1, A_2… A_n are propositions such that $a_1 + a_2 + \ldots + a_n \ge r \underset{\square}{\mathbf{1}} + s\, a_0$, where a_i is the characteristic function of A_i and denotes the unit function. (what this means is that, without knowing which ones they might be, one can be sure that at least r – and if A_0 holds then at least r + s – of the propositions, $A_1, A_2, \ldots, A_n$ are bound to be true.) Then,

$C_h(a_1) + \ldots\ldots + C_h(a_n) \ge r + s\, C_h(a_0)$

To justify (6) we observe that if the person pays the agent ₹ ($C_h(a_1) + \ldots + C_h(a_n)$) then he will assert the propositions $A_1, A_2 \ldots A_n$. In view of the hypothesis of the theorem this means that, when these propositions are tested, he will be obliged to pay me back ₹ r in any case and ₹ (r + s) if in fact A_0 is true. This shows that for a fee of ₹ ($\sum_1^n C_h(A_{i-r})$) the agent is willing (among other things) to commit himself to pay ₹ s if A_0 is true; and it follows that

$$s\, C_h(A_0) \le \sum_1^n C_h(A_i) - r.$$

Characterization of a Chance Function

We have observed that the chance function of a rational agent has property (6). Here we shall show (corollary 1 to theorem 4) that every function with this property is the chance function of some rational agent. Anticipating this result, we define a chance function to be any function with this property.

Definition 4

A chance function is a map C_h: Ɓ $\rightarrow [0, 1]$ with the property: if n, r, s are non-negative integers and $A_0, A_1, A_2 \ldots A_n$ are propositions such that $a_1 + a_2 + \ldots\ldots\ldots + a_n \geq r \frac{1}{\square} + s\, a_0$ then $C_h(A_1) + \ldots\ldots\ldots\ldots + C_h(A_n) \geq r + s\, C_h(A_0)$.

Theorem 2

Every chance function has properties (1) to (5) of section “Properties of Chance Functions” viz:

(a) $C_h(F) = 0,\ C_h(\tau) = 1$.
(b) $C_h(A) \leq C_h(B)$ whenever $A \Longrightarrow B$.
(c) $\max\left(C_h(A), C_h(B)\right) \leq C_h(A \cup B) \leq C_h(A) + C_h(B)$.
(d) If $A \cup B = \tau$ then $C_h(A) + C_h(B) \geq 1$.
(e) $C_h(A) + C_h(A^c) \geq 1$.

Proof

(a) Putting $n = r = s = 1$, $a_1 = \tau$, $a_0 = F$, we get $C_h(\tau) \geq 1 + C_h(F)$.
Thus, $C_h(\tau) - C_h(F) \geq 1$ and the only possibility is $C_h(\tau) = 1;\ C_h(F) = 0$.

(b) Putting $r = 0$; $n = s = 1$; $A_1 = B$, $A_0 = A$ we get the desired inequality.

(c) The first inequality follows from (b) and for the second we put $n = 2$, $r = 0$, $s = 1$, $A_1 = A$, $A_2 = B$, $A_0 = A \cup B$.

(d) and (e) are corollaries of (c).

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